

Extended Weyl-Heisenberg algebra and Rubakov-Spiridonov superalgebra: Anyonic realizations.

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Abstract

We give the realizations of the extended Weyl-Heisenberg (WH) algebra and the Rubakov-Spiridonov (RS) superalgebra in terms of anyons, characterized by the statistical parameter $\nu \in [0, 1]$, on two-dimensional lattice. The construction uses anyons defined from usual fermionic oscillators (Lerda-Sciuto construction). The anyonic realization of the superalgebra $sl(1/1)$ is also presented.

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1 Introduction

Anyons are particles with any statistics that interpolate between bosons and fermions [1, 2, 3]. They exist only in two dimension because the configuration space of collection of identical particles has some special topological properties allowing arbitrary statistics. On the other hand supersymmetry (underlining Z_2 -graded superalgebras) provided us with an elegant symmetry between fermions and bosons [4, 5]. Due to his success, there have been many variances to generalize its structures to incorporate other kinds of statistics. The first attempt in this sens was proposed by Rubakov and Spiridonov by combining bosonic and parafermionic degrees of freedom leading to the so-called para-supersymmetric quantum mechanics of order 3 [6]. The 3-fractional Rubakov-Spiridonov superalgebra has been extended to an arbitrary order k by Khare [7] generalizing the Z_2 by a Z_k -grading. The formalism of parasupersymmetry involves a bosonic degree of freedom (described by a complex variable) and a parafermionic degree of freedom (described by a generalized Grassmann variable of order k [8, 9]). In other words, to pass from a Z_2 -graded theory to a Z_k -graded one, we retain the bosonic variable and replace the fermionic variable by a parafermionic one. We note that the k -fermionic variables [10, 11] have been also used to extend the Rubakov-Spiridonov algebra [12]. Fractional supersymmetry was also developed without any explicit introduction of parafermionic or k -fermionic degrees of freedom. In this respect, fractional supersymmetry was worked by Quesne and Vansteenkiste [13] owing to the introduction of the extended Weyl-Heisenberg algebra [14] (called also extended oscillator algebra).

The connection between anyons and fractional supersymmetry seems apparently two very distinct subjects and have not been considered previously in the literature. However, as we will see the statistical parameter of anyons and the order of the fractional supersymmetry are deeply related. This connection is based on the anyonization of the extended Weyl-Heisenberg algebra. Then, considering the creation and annihilation anyonic operators (called also anyonic oscillators), constructed by Lerda and Sciuto on a two dimensional lattice [15], we discuss the anyonic realization of the extended Weyl-Heisenberg algebra. The latter realization will be the cornerstone to provide an anyonic realization of the fractional Rubakov-Spiridonov superalgebra.

The present letter is organized as follows. First, we recall basis notions connected with the Lerda-Sciuto anyonic oscillators on a two dimensional Lattice. In section 3, we show the usefulness of the anyonic creation and annihilation operators to provide a new realization of the extended Weyl-Heisenberg algebra involving objects with fractional spin. Section 4 is devoted to derivation of the fractional Rubakov-Spiridonov superalgebra using the generators of the Weyl-Heisenberg algebra. In section 5, we show that the anyons can be also used to realize the superalgebra $sl(1/1)$ which is undeformed in the quantum superalgebra context [16, 17, 18] and not fractionnal in the Rubakov-Spiridonov language.

2 Basics tools: Lerda-Sciuto anyonic oscillators

In this section, we recall necessary minimum of details concerning anyonic oscillators (see the reference [15, 16]) on $2d$ square lattice Ω with spacing $a = 1$. We give a two-

component fermionic spinor field by

$$S^- = \begin{pmatrix} s_1^-(x) \\ s_2^-(x) \end{pmatrix}, \quad (1)$$

and its conjugate hermitian by

$$S^+ = (s_1^+(x), s_2^+(x)), \quad (2)$$

such that the components of these fields satisfy the following standard anticommutation relations

$$\begin{aligned} \{s_i^-(x), s_j^-(y)\} &= 0 \\ \{s_i^+(x), s_j^+(y)\} &= 0 \\ \{s_i^-(x), s_j^+(y)\} &= \delta_{ij}\delta(x, y), \end{aligned} \quad (3)$$

for $i, j \in \{1, 2\}$ and $x, y \in \Omega$. Here, $\delta(x, y)$ is the conventional lattice δ -function: $\delta(x, y) = 1$ if $x = y$ and vanishes if $x \neq y$. We use the definition of the lattice angle functions $\Theta_{\pm\Gamma_x}(x, y)$ as recited in [16]; where Γ_x is the curve associated to each site $x \in \Omega$ and the signs \pm indicate the two kinds of rotation direction on Ω . This definition can be identified to that is discussed in [15] by Lerda and Sciuto.

The expression of anyonic oscillators are given in term of fermionic spinors and angle functions as follows

$$\begin{aligned} a_i^-(x_{\pm}) &= e^{i\nu\Delta_i(x_{\pm})} s_i^-(x) \\ a_i^+(x_{\pm}) &= s_i^+(x) e^{-i\nu\Delta_i(x_{\pm})}. \end{aligned} \quad (4)$$

The number ν appearing in this equation is usually called statistics parameter. The elements $\Delta_i(x_{\pm})$ are given by

$$\Delta_i(x_{\pm}) = \sum_{y \in \Omega} s_i^+(x) \Theta_{\pm\Gamma_x}(x, y) s_i^-(y), \quad (5)$$

which satisfy the following commutation relations

$$\begin{aligned} [\Delta_i(x_{\pm}), s_j^-(y)] &= -\delta_{ij} \Theta_{\pm\Gamma_x}(x, y) s_i^-(y) \\ [\Delta_i(x_{\pm}), s_j^+(y)] &= \delta_{ij} \Theta_{\pm\Gamma_x}(x, y) s_i^+(y) \\ [\Delta_i(x_{\pm}), \Delta_j(y_{\pm})] &= 0 \end{aligned}$$

Now, we can show that the anyonic oscillators satisfy the following algebraic relations

$$\begin{aligned} [a_i^-(x_{\pm}), a_i^-(y_{\pm})]_{\Lambda^{\mp}} &= 0, & x > y \\ [a_i^-(x_{\pm}), a_i^+(y_{\pm})]_{\Lambda^{\pm}} &= 0, & x > y \\ [a_i^+(x_{\pm}), a_i^-(y_{\pm})]_{\Lambda^{\pm}} &= 0, & x > y \\ [a_i^+(x_{\pm}), a_i^+(y_{\pm})]_{\Lambda^{\mp}} &= 0, & x > y \\ [a_i^-(x_{\pm}), a_i^+(x_{\pm})] &= 1, \\ [a_i^-(x_{\pm}), a_j^+(y_{\pm})] &= 0, & i \neq j \\ [a_i^+(x_{\pm}), a_j^-(y_{\pm})] &= 0, & i \neq j \\ [a_i^+(x_{\pm}), a_j^+(y_{\pm})] &= 0, & i \neq j \\ [a_i^{\pm}(x_{-}), a_j^{\pm}(y_{+})] &= 0, & \forall i, j \\ [a_i^-(x_{-}), a_j^+(y_{+})] &= \delta_{ij} \delta(x, y) \Lambda^{+[\sum_{z < x} - \sum_{z > x}] s_j^+(z) s_j^-(z)}, \\ [a_i^-(x_{-}), a_j^+(y_{+})] &= \delta_{ij} \delta(x, y) \Lambda^{-[\sum_{z < x} - \sum_{z > x}] s_j^+(z) s_j^-(z)}. \end{aligned} \quad (6)$$

where $\Lambda^\pm = e^{\pm i\nu\pi}$, $[X, Y]_\Lambda = XY + \Lambda YX$ and

$$x > y \Leftrightarrow \begin{cases} x_+ > y_+ \Leftrightarrow \begin{cases} x_2 > y_2 \\ x_1 > y_1, x_2 = x_1 \end{cases} \\ x_- < y_- \Leftrightarrow \begin{cases} x_2 < y_2 \\ x_1 < y_1, x_1 = x_2 \end{cases} \end{cases}$$

One obtains also

$$(a_i^\pm(x_\pm))^2 = 0, \quad (7)$$

which is known as the hard core condition.

We would like to stress that despite the many formal analogies, the anyonic oscillators does not have anything to do with the k -fermions, q -deformed bosons with $q = e^{2\pi i/k}$, for several reasons: (i) the k -fermions can be defined in any dimensions whereas the anyons are strictly two-dimensional objects, (ii) the anyons are non-local contrarily to the k -fermions. The latter objects constitute a mathematical tool, introduced in the context of quantum algebras, which was used to go beyond the conventional statistics in any dimension and taking into account some perturbation (deformation) responsible of small deviations from the FD and BE usual statistics [19, 20].

3 Anyonic realization of the extended Weyl-Heisenberg algebra

For fixed $\nu \in [0, 1]$, the extended Weyl-Heisenberg algebra is defined as an algebra generated by the operators a_+ , $a_- = (a_+)^+$, N and $K = (K^+)^{-1}$ satisfying the following relations [12, 13]

$$[a_-, a_+] = \sum_{s=0}^{\frac{2}{\nu}-1} f_s P_s, \quad [N, a_\pm] = \pm a^\pm \quad (8)$$

$$K a_+ = e^{i\pi\nu} a_+ K, \quad K a_- = e^{-i\pi\nu} a_- K$$

$$[K, N] = 0, \quad K^{\frac{2}{\nu}} = 1.$$

Here, f_s are some real parameters and the admissible values of the statistical parameter is restricted by the condition $\frac{2}{\nu} \in \{2, 3, 4, \dots\}$. The operators P_s are polynomials in the grading operator K defined by

$$P_s = \frac{\nu}{2} \sum_{t=0}^{\frac{2}{\nu}-1} e^{-i\pi\nu st} K^t, \quad (9)$$

for $s = 0, 1, 2, \dots, \frac{2}{\nu} - 1$. One can see that the P_s 's operators satisfy the following relations

$$\sum_{t=0}^{\frac{2}{\nu}-1} P_s = 1, \quad P_s P_t = \delta(s, t) P_s \quad (10)$$

where δ is the Kronecker symbol. Furthermore, these operators satisfy

$$P_s a^+ = a^+ P_{s-1}, \quad a^- P_s = P_{s-1} a^-. \quad (11)$$

Note that the equation (9) can be conversed in the form

$$K^t = \sum_{s=0}^{\frac{2}{\nu}-1} e^{i\pi\nu st} P_s, \quad (12)$$

with $t = 0, 1, \dots, \frac{2}{\nu} - 1$. It is clear that the commutation relation between a_- and a_+ (equations (8)) can be written as

$$[a_-, a_+] = \sum_{s=0}^{\frac{2}{\nu}-1} c_s K^s \quad (13)$$

where the c_s are related to parameters f_t by

$$f_t = \sum_{s=0}^{\frac{2}{\nu}-1} e^{i\pi\nu st} c_s \quad (14)$$

or conversely by

$$c_s = \frac{\nu}{2} \sum_{t=0}^{\frac{2}{\nu}-1} f_t e^{-i\pi\nu st}. \quad (15)$$

To show that the extended Weyl-Heisenberg algebra can be realized by means of anyonic oscillators of statistics ν in a quite direct though non-trivial way, we start by introducing the local operators

$$\begin{aligned} a_+(x) &= a_1^+(x_+) a_2^-(x_+) \\ a_-(x) &= a_2^+(x_-) a_1^-(x_-) \end{aligned} \quad (16)$$

and

$$N(x) = a_1^+(x_+) a_1^-(x_+) - a_2^+(x_+) a_2^-(x_+). \quad (17)$$

It is not difficult to verify, from the commutation relations (6), that

$$\begin{aligned} a_+(y) a_+(x) &= e^{i2\pi\nu} a_+(x) a_+(y) \\ a_-(y) a_-(x) &= e^{-i2\pi\nu} a_-(x) a_-(y) \end{aligned} \quad (18)$$

for $x > y$. The latter equation show that the local operators defined by (16) have braiding properties and one have to specify the ordering of the points x and y if we change the braiding orientation. In the same way, with a straighfoward application of equations (6), we get

$$\begin{aligned} [N(x), a_{\pm}(y)] &= \pm a_{\pm}(x) \delta(x, y), \\ [a_+(x), a_-(y)] &= 0, \quad x \neq y \end{aligned} \quad (19)$$

The commutation relation of $a_+(x)$ and $a_-(x)$ (i. e. at the same point) is slightly more complicated. However, by an adequate using of the commutation relations of the anyonic oscillators, it is not hard to show that

$$[a_+(x), a_-(x)] = \prod_{y < x} e^{-i\nu\pi N(y)} N(x) \prod_{z > x} e^{-i\nu\pi N(z)}. \quad (20)$$

Now, we introduce the global creation, annihilation and number anyonic operators of spin $s = \frac{\nu}{2}$ on the lattice in terms of the local anyonic ones. They are defined by

$$\begin{aligned} a_+ &= \sum_{x \in \Omega} a_1^+(x_+) a_2^-(x_+) \\ a_- &= \sum_{x \in \Omega} a_2^+(x_-) a_1^-(x_-), \end{aligned} \quad (21)$$

and

$$N = \sum_{x \in \Omega} (a_1^+(x_\pm) a_1^-(x_\pm) - a_2^+(x_\pm) a_2^-(x_\pm)) \quad (22)$$

The grading operator (known also as Klein operator) is defined by

$$K = e^{[i\pi\nu \sum_{x \in \Omega} (a_1^+(x_\pm) a_1^-(x_\pm) - a_2^+(x_\pm) a_2^-(x_\pm))]} \quad (23)$$

It is clear that the operator K satisfy

$$K^{\frac{2}{\nu}} = 1 \quad (24)$$

and we have also

$$(a_+)^+ = a_-. \quad (25)$$

The operators N and K are commuting. Furthermore using the structure relations of the anyonic oscillators on the lattice Ω , we have the following commutation relations

$$[a_+, a_-] = \sum_{x \in \Omega} \left(\prod_{y < x} e^{[-i\pi\nu N(y)]} \right) N(x) \left(\prod_{z > x} e^{[-i\pi\nu N(z)]} \right), \quad (26)$$

$$[N, a_\pm] = \pm a_\pm, \quad K a_\pm = e^{\pm i\nu\pi} a_\pm K.$$

To write the commutation relation involving a_- and a_+ (Eqs (26)), in a form similar to the one given in equations (8), we show that

$$\begin{aligned} \sum_{x \in \Omega} \left(\prod_{y < x} e^{[-i\pi\nu N(y)]} \right) N(x) \left(\prod_{z > x} e^{[-i\pi\nu N(z)]} \right) &= - \sum_{t=0}^{\frac{2}{\nu}-1} f_t P_t \\ &= - \sum_{t=0}^{\frac{2}{\nu}-1} c_s K^s \end{aligned} \quad (27)$$

where the parameters f_t and c_t are given by

$$f_t = - \frac{\sin(2\pi\nu t)}{\sin(\pi\nu)} \quad (28)$$

and

$$c_s = \begin{cases} 0, & s \neq 2, \frac{2}{\nu} - 2 \\ \frac{1}{2\sin(\pi\nu)}, & s = 2 \\ -\frac{1}{2\sin(\pi\nu)}, & s = \frac{2}{\nu} - 2 \end{cases} \quad (29)$$

To obtain this result, we followed a similar method that one established by Lerda and Sciuto in [15]. The first step of this method is based on the fact that the local operator

$N(x)$ admits only the eigenvalues 0 and +1 for any $x \in \Omega$ according to the Pauli exclusion principle for anyon operators (hard core condition). Then, one have the identity

$$N(x) = \frac{\sin(\pi\nu N(x))}{\sin(\pi\nu)} \quad (30)$$

for any site $x \in \Omega$. Reducing the lattice Ω to only one site, x_0 for instance, we have $N = N(x_0)$ and thus the equality (27) hold thanks to equation (30).

Assuming that the equation (27) is valid for a lattice of n sites $\{x_i, i = 1, \dots, n\}$, we add an extra point x_{n+1} . For the lattice with $(n+1)$ sites, the equation (27) becomes

$$\sum_{i=1}^n \left(\prod_{j<i} e^{-i\pi\nu N(x_j)} N(x_i) \prod_{k>i} e^{-i\pi\nu N(x_k)} \right) e^{N(x_{n+1})} + e^{i\pi\nu N(x_{n+1})} N(x_{n+1}) = \frac{\sin(\pi\nu N)}{\sin(\pi\nu)} \quad (31)$$

where $N = \sum_{i=1}^{n+1} N(x_i)$. The last equation is easily designed using the equation (26) and (30). In view of the relations (24), (26) and (27), we conclude that the operators a_+ , a_- , N and K close the extended Weyl-Heisenberg algebra.

The operators $a_-^{\frac{2}{\nu}}$, $a_+^{\frac{2}{\nu}}$ and $K^{\frac{2}{\nu}}$ belong the centre of the extended Weyl-Heisenberg algebra. It is straightforward to prove also that the operator

$$C = a_- a_+ + \frac{\sin(\pi\nu(2N+1))}{2\sin^2(\pi\nu)} \quad (32)$$

is an invariant of the WH algebra. Then, the extended WH algebra admits finite-dimensional representations of dimension $\frac{2}{\nu}$ such that

$$a_-^{\frac{2}{\nu}} = \alpha \mathbf{1}_{\frac{2}{\nu} \times \frac{2}{\nu}} \quad (33)$$

and

$$a_+^{\frac{2}{\nu}} = \beta \mathbf{1}_{\frac{2}{\nu} \times \frac{2}{\nu}} \quad (34)$$

where $(\alpha, \beta) \in \mathbf{C}^2$. Three types of representation can be considered: (i) $\alpha = \beta = 0$ (nilpotent representation), (ii) $\alpha = \beta = 1$ (cyclic or periodic representations) and (iii) $\alpha = 0$ and $\beta \neq 0$ or $\alpha \neq 0$ and $\beta = 0$ (semi-periodic representations). In a representation of type (i), the creation a_+ and annihilation a_- seems to be similar to ones of $(k = \frac{2}{\nu})$ -fermions [11, 21, 22] which satisfies the generalized exclusion principle according to which no more than $(k-1)$ particles can live in the same quantum state [20].

4 Fractional RS superalgebra through fermionic anyons

In view of the fact that supersymmetry (Models and symmetries with Z_2 -grading) provide symmetry between bosons and fermions, it is natural to enquire if one could generalize these structures by including the exotic statistics. More than ten years back [6], Rubakov

and Spiridonov discussed such generalizations. In particular, they constructed a superalgebra defined by the following structure relations

$$\begin{aligned}
(E_-)^3 &= 0 \\
[E_-, H] &= 0 \\
(E_-)^2 E_+ + E_- E_+ E_- + E_+ (E_-)^2 &= 2E_- H
\end{aligned} \tag{35}$$

and the hermitian conjugated relations. The RS superalgebra generated by $\{E_+, E_-, H\}$ was realized as Z_3 -graded symmetry between one boson and one parafermion of order 2. The generalization of RS superalgebra (37) for an arbitrary order k is given by [7].

$$\begin{aligned}
(E_-)^k &= 0 \\
[E_-, H] &= 0 \\
(E_-)^{k-1} E_+ + (E_-)^{k-2} E_+ E_- + E_+ (E_-)^{k-1} &= (k-1)(E_-)^{k-2} H
\end{aligned} \tag{36}$$

together with their hermitian conjugates. The realization of the RS superalgebra (36) involve one boson and $(k-1)$ parafermions. Note that for $k=2$, the relations (36) reduces to ones defining the superalgebra $sl(1/1)$. Then the RS superalgebra seems to be a k -fractional extension of the superalgebra $sl(1/1)$.

In this part of our work, we will discuss how using the generators of extended Weyl-Heisenberg algebra (remember that the operators of this algebra are defined in terms of anyonic oscillators living on two-dimensional lattice) one can give an anyonic realization of the so-called Rubakov-Spiridonov superalgebra. Indeed, By means of the operators P_i 's (polynomials in the grading operator K) and the creation and annihilation operators a_+ and a_- (that are defined by coupling two 2d-lattice anyons), we are thus in position to define the operators

$$E_- = a_-(1 - P_{k-1}), \quad E_+ = a_+(1 - P_0) \tag{37}$$

among k possible definitions. The $(k-1)$ other definitions can be obtained by a simple permutation of indices $0, 1, 2, 3, \dots, k-1$. Note that the operators E_+ and E_- are defined such that satisfy also

$$E_+ = (E_-)^+ \tag{38}$$

As a first result, one can show, using the structure relations of the extended Weyl-Heisenberg algebra introduced in the previous section, that

$$\begin{aligned}
(E_-)^m &= (a_-)^m [1 - P_{k-m} + P_{k-m+1} + \dots + P_{k-1}] \\
(E_+)^m &= (a_+)^m [1 - P_0 + P_1 + \dots + P_{m-1}]
\end{aligned} \tag{39}$$

for $m = 1, 2, \dots, k-1$. For the particular case $m = k$, we have

$$(E_-)^k = 0, \quad (E_+)^k = 0. \tag{40}$$

Let us note that for $k=2$, we have the nilpotency condition of the $sl(1/1)$ generators E_- and E_+ . We continue with the construction of the fractional Rubakov-Spiridonov

superalgebra. So, one can obtain also, by some manipulations more or less complicated, the following identity

$$(E_-)^m E_+ = (a_-)^m a_+ (P_1 + P_2 + \dots + P_{k-m}) \quad (41)$$

which leads to the relations

$$(E_-)^m E_+ (E_-)^l = (a_-)^m a_+ (a_-)^l, \quad m + l = k - 1, \quad (42)$$

with $m \neq 0$; $l \neq k - 1$ and $m \neq k - 1$; $l \neq 0$,

$$E_+ (E_-)^{k-1} = a_+ (a_-)^{k-1} P_0 \quad (43)$$

$$(E_-)^{k-1} E_+ = (a_-)^{k-1} a_+ P_1. \quad (44)$$

The equations (42), (43) and (44) are the basic ones to get the multilinear relation which should be satisfied by the generators E_- and E_+ of the Rubakov-Spiridonov superalgebra. Indeed, introducing an even operator H , we show that

$$\sum_{i=0}^{k-1} (E_-)^{k-1-i} E_+ (E_-)^i = (k-1) (E_-)^{k-2} H, \quad (45)$$

where H is defined by

$$H = \sum_{s=0}^{k-1} H_{k-s} P_s \quad (46)$$

where

$$H_k = a_+ a_- - \frac{1}{k-1} \sum_{s=2}^{k-1} \sum_{t=1}^{s-1} h(N-t) \quad (47)$$

and

$$H_{k-s} = H_k + \frac{1}{k-1} \sum_{t=0}^{k-2} \sum_{i=1}^s h(N+i-1-t), \quad s \neq 0. \quad (48)$$

The function h is given by

$$h(N-l) = \frac{\sin(\nu\pi(N-l))}{\sin(\nu\pi)} \quad (49)$$

The operator H commute with the generators E_+ and E_-

$$[H, E_{\pm}] = 0. \quad (50)$$

The nilpotency relations (40) together with the multilinear relation (45) and the commutation relation (50) close the fractional Rubakov-Spiridonov superalgebra. Let us note that this superalgebra is different from one obtained in the context of quantum superalgebra which is characterized by the following structure relations [17, 18]

$$\begin{aligned} \{E_+, E_-\} &= \frac{q^H - q^{-H}}{q - q^{-1}}, \\ [E_{\pm}, H] &= 0, \\ (E_+)^2 &= 0, \quad (E_-)^2 = 0. \end{aligned} \quad (51)$$

Two limiting cases $\nu = 1$ and $\nu = 0$ are interesting. In the case where $\nu = 1$, it is easy to see the Rubakov-Spiridonov superalgebra gives the well known superalgebra $sl(1/1)$. In

the case $\nu = 0$, the operator K is equal to unit operator and the projection operators P_i 's vanishes. Then, we have

$$E_+ = a_+, \quad E_- = a_- \quad (52)$$

where the operator a_+ and a_- are defined in terms of two bosonic oscillators (because in the limiting case $\nu = 0$, the anyons becomes bosons). As consequence, E_+ , E_- and H generate the $sl(2)$ algebra. Then, in the limit $\nu = 1$, the RS superalgebra reduces to Z_2 -graded $sl(1/1)$ superalgebra and for $\nu = 0$, we have $sl(2)$ (no graduation).

We note finally that the structure relations defining the RS superalgebra are analogous to those defining the so-called fractional supersymmetric quantum mechanics [6, 7]. So, one can ask about the physical meaning of the even operator H and if one can interpret it as an hamiltonian describing a system of two anyons constrained to evolve in a two dimensional lattice. We believe that this point merit more investigation and can help us to learn more about the physics in low dimensions.

5 Anyonic realization of the $sl(1/1)$ superalgebra

In this last section, we would draw the attention that the superalgebra $sl(1/1)$ (undeformed in the quantum superalgebra context and no fractional in the Rubakov-Spiridonov sens) can be realized using two anyons living on two dimensional lattice. In this order define the operators

$$\begin{aligned} q_{i,+} &= a_+ P_{k-i} \\ q_{i,-} &= P_{k-i} a_- \end{aligned} \quad (53)$$

and

$$h_i = a_- a_+ P_{k-i} + a_+ a_- P_{k-i+1} \quad (54)$$

(for $i = 2, 3, \dots, k$) where the operators a_+ and a_- are defined by (21) in terms of the Lerda-Sciuto anyonic oscillators. A direct computation show that operators $q_{i,+}$, $q_{i,-}$ and h_i satisfy the structure relations

$$\begin{aligned} [q_{i,\pm}, h_i] &= 0, \quad \{q_{i,-}, q_{i,+}\} = h_i \\ (q_{i,+})^2 &= 0, \quad (q_{i,-})^2 = 0 \end{aligned} \quad (55)$$

that are ones defining the $sl(1/1)$ superalgebra. This result show how one can realize the (undeformed) superalgebra $sl(1/1)$ using the anyonic oscillators. Furthermore, our realization is new in the sens that not has been considered previously in the literature and should not be confused with Frappat et al [18] (see also the references therein) realizations which concerns the anyonic representations of the quantum (deformed) superalgebras. One can think that the realization obtained here of $sl(1/1)$ superalgebra, which is essentially a coupling the 2d lattice anyons, suggests some probable manifestation of the supersymmetry in low dimensional physics.

It is important to note that the odd operators E_+ and E_- (see Eqs (37)) can be expressed as sums of the odd operators $q_{i,+}$ and $q_{i,-}$, generating (with h_i) the superalgebra $sl(1/1)$, as follows

$$E_+ = \sum_{i=2}^k q_{i,+}, \quad E_- = \sum_{i=2}^k q_{i,-} \quad (56)$$

which are Ladder operators of the fractional Rubakov-Spiridonov superalgebra (see the previous section).

6 Conclusion

In the present letter, we introduced the anyonic realization of the extended Weyl-Heisenberg algebra by means of Lerda-Sciuto anyonic oscillators of statistics ν on two dimensional lattice. We discussed how the Z_ν -grading of this algebra leads to the realization of the fractional Rubakov-Spiridonov superalgebra. We then proved the connection between the fractional spin of anyons and the order of the RS superalgebra. We showed also that the superalgebra $sl(1/1)$ can be constructed following an anyonic scheme.

It is clear that remain many open problems for future study. One of them would be a better understanding of the physical meaning of the operator $H(\text{Eq}(46))$ and if can it be related to the hamiltonian describing planar systems evolving under topological Chern-Simon's interaction. Another would concerns the anyonic realizations of the fractional analogous of the other superalgebras (undeformed in the quantum groups language) than $sl(1/1)$. These problems are under study and we hope to report on them in the near future.

Acknowledgments

The authors would like to thank the Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, for its warm hospitality. The author M. D would like to express his sincere thanks to Prof. Yu Lu (head of Condensed Matter in ICTP) for his kind invitation. The author J. D would acknowledges the Max-Planck-Institut für Physik Komplexer Systeme for link hospitality during the stage in which one part of this paper was done, and She particularly would like to thank the associateship Scheme in the Abdus Salam ICTP for its facilities.

References

- [1] H. M. Leinaas and J. Myrheim, Nuovo Cimento, **B37**, 1 (1977).
- [2] O. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 975 (1982).
- [3] G. A. Goldin, R. Menikoff and D. H. Sharp, J. Math. Phys. **21**, 650 (1980); **22**, 1664 (1981).
- [4] E. Witten, Nucl. Phys. B **138**, 513 (1981).
- [5] S. Ferrara, Supersymmetry, (World Scientific, Singapore, 1987).
- [6] V. A. Rubakov and V. P. Spiridonov, Mod. Phys. Lett. **3**, 1337 (1988).
- [7] A. Khare, J. Phys. A **25**, L749 (1992); J. Math. Phys **34**, 1277 (1993).
- [8] S. Majid and M. J. Rodriguez-Plaza, J. Math. Phys. **35**, 3753 (1994).
- [9] J. A. de Azcárraga and J. Macfarlane, J. Math. Phys. **37**, 1115 (1996).

- [10] M. Daoud, Y. Hassouni and M. Kibler, *Yad. Fiz.* **61**, 1935 (1998).
- [11] M. Daoud, Y. Hassouni and M. Kibler, in *Symmetries in Science X*, Eds. B. Gruber and M. Ramek (Plenum, New York, 1998).
- [12] M. Daoud and M. Kibler, in *Symmetry and Structural properties of Condensed Matter*, eds. T. Lulek, B. Lulek and A. Wal (World Scientific, Singapore, 2001).
- [13] C. Quesne and N. Vanteenkiste, *Phys. Lett. A* **240**, 21 (1998).
- [14] M. S. Plyushchay, *Ann. Phys. (N. Y.)* **245**, 339 (1996).
- [15] A. Lerda and S. Sciuto, *Nucl. Phys. B* **401**, 613 (1993).
- [16] M. Daoud, J. Douari and Y. Hassouni, *acta physica slovacica* **49**, 945 (1999).
- [17] R. Floreanini, P. Spiridonov and L. Vinet, *Phys. Lett. B* **242**, 383 (1990); *Commun. Math. Phys* **137**, 149 (1991).
- [18] L. Frappat, S. Sciuto, A. Sciarrino and P. Sorba, *Phys. Lett. B* **369**, 313 (1996); *J. Phys. A* **30**, 903 (1997).
- [19] M. Daoud and M. Kibler, *Phys. Lett. A* **206**, 13 (1995).
- [20] M. Daoud and Y. Hassouni, *Helv. Phys. Acta* **71** 599 (1998).
- [21] M. Mansour, M. Daoud and Y. Hassouni, *Phys. Lett. B* **454**, 281 (1999); *Rep. Math. Phys* **44**, 435 (1999).
- [22] A. Jellal, M. Daoud and Y. Hassouni, *Phys. Lett. B* **474**, 122 (2000).